Int. J. Solids Structures, 1969, Vol. 5, pp. 351 to 367. Pergamon Press. Printed in Great Britain

THERMAL STRESSES IN A SOLID WEAKENED BY AN EXTERNAL CIRCULAR CRACK*

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Abstract—Linear thermoelastic problems are solved for the thermal stress and displacement fields in an elastic solid of infinite extent weakened by a plane of discontinuity or crack occupying the space outside of a circular region. The faces of the crack are heated by maintaining them at certain temperature and/or by some prescribed heat flux the distributions of which are such that their magnitudes diminish at infinity. Special emphasis is given to the case when the circular region surrounded by the external crack is insulated from heat flow. The solution to this thermal stress problem may be combined with that of applying appropriate tractions to the crack faces, thus providing the necessary ingredients for extending the Dugdale hypothesis to thermally-stressed bodies containing cracks. More specifically, the results of the analysis permit an estimate of the plastic zone size and the plastic energy dissipation for an external circular crack. Information of this kind contributes to the understanding of the mechanics of fracture initiation in ductile materials.

INTRODUCTION

PREVIOUS efforts on steady-state thermoelastic problems have been focused mainly on problems dealing with bars, plates and cylinders. A complete account of these developments is clearly beyond the scope of this article. On the other hand, systematic study of the effect of plane cracks on thermal stresses set up in an elastic solid has a quite recent history started in the past few years.

Beginning with the work of Olesiak and Sneddon [1], the method of dual integral equations in the Hankel transforms was used to determine the distribution of temperature and stress in a solid containing a penny-shaped crack across whose surfaces there is a prescribed flux of heat. By having the same thermal conditions on the upper and lower faces of the crack, the problem was reduced to one of specifying certain mixed boundary conditions on a semi-infinite solid. The case of heat supplied antisymmetrically with respect to the crack plane was treated by Florence and Goodier [2]. Using potential function theory, Kassir and Sih [3] presented explicit solutions to a class of three-dimensional thermal stress problems with an elliptical crack whose faces are thermally disturbed by both symmetric and antisymmetric temperatures and/or temperature gradients. Their results include those in [1, 2] as limiting cases. Further, Kassir and Sih [3] showed that for any small region around the outer boundary of an elliptically-shaped crack the thermal stresses and displacements correspond to a situation which is locally one of plane strain as derived earlier by Sih [4] using the equations of two-dimensional thermoelasticity.

* The research work presented in this paper was obtained in the course of an investigation conducted under Grant NGR-39-007-025 with the National Aeronautics and Space Administration.

This investigation presents an analysis of the steady-state axisymmetric thermoelastic problem concerning two semi-infinite solids joined over a circular region. The unconnected portion of the solids may be regarded as an external penny-shaped crack. Thermal boundary conditions are standard in that the temperature or heat flux must be known at the surfaces of the crack in such a way that the temperature distribution in the solid is determined uniquely. With this temperature distribution known, introduction of a thermoelastic potential reduces the problem to one in axisymmetric isothermal elasticity with body forces.

The circular region connecting the two semi-infinite solids is assumed to be insulated* from heat flow, while the crack surface is heated by temperature $T(r)$ that may vary as a function of the radial distance r from the center of the circular region of unit radius. Two special cases are considered in detail. In the first case, $T(r)$ is a constant prescribed over an annular region surrounding the circle $r = 1$. In the second case, it is assumed that the function $T(r)$ varies according to r^{-n}, where $n > 1$. The problem in which the crack surface is heated by some flux of heat may be solved in the same fashion.

Another objective of this work is to calculate the stress–intensity factors [5] the critical values of which control the onset of crack propagation in brittle materials. For ductile materials, the Dugdale hypothesis [6] may be adopted by assuming that the plastic zone developed at the crack border can be approximated by a very thin layer in the form of a ring. An estimate of the plastic energy dissipation of the crack can also be obtained from the results presented in this paper.

AXISYMMETRIC EQUATIONS OF THERMOELASTICITY

Let an external penny-shaped crack be situated in the plane $z = 0$ and be opened out by the application of heat to its surfaces such that the deformation is symmetrical about the z-axis. Referring to cylindrical coordinates (r, θ, z), the stress components are independent of the angle θ, and all derivatives with respect to θ vanish. The components of the displacement vector $\mathbf{u}$ for axially symmetrical deformation are (u, o, w), and the non-vanishing components of the stress tensor $\boldsymbol{\sigma}$ will be denoted by σ_r, σ_θ, σ_z and τ_{rz}.

If the heat flux vector does not depend on the components of strain, then the displacement equations of equilibrium become

$$
\begin{aligned}
2(1-\nu)\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}\right)+(1-2\nu)\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 w}{\partial r\,\partial z} &= 2(1+\nu)\alpha\frac{\partial T}{\partial r},\\
(1-2\nu)\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right)+2(1-\nu)\frac{\partial^2 w}{\partial z^2}+\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right) &= 2(1+\nu)\alpha\frac{\partial T}{\partial z},
\end{aligned}
\tag{1}
$$

and can be solved independently from the equation of steady-state heat conduction

$$\nabla^2 T(r, z) = 0. \tag{2}$$

Here, T is the temperature increase referred to some reference state and ∇^2 stands for

$$\nabla^2 = \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}.$$

* No additional difficulties are encountered if heat is allowed to flow through the circular region. Alterations in the thermal boundary conditions are discussed in the Appendix.

In equations (1), α is the coefficient of linear expansion and ν is Poisson's ratio of the material.

When both the mechanical and thermal properties of the solid are assumed to be isotropic and homogeneous, the stress components may be obtained from the displacement components by means of the Duhamel–Neumann law, which in dyadic notation takes the form

$$\boldsymbol{\sigma} = \mu\left\{\nabla\mathbf{u}+\mathbf{u}\nabla+\frac{2}{1-2\nu}[\nu\nabla\,.\,\mathbf{u}-(1+\nu)\alpha T]\mathbf{I}\right\}, \tag{3}$$

in which μ is the shear modulus, $\mathbf{I}$ the unit dyad and ∇ the usual del operator.

Kassir and Sih [3]* have shown that the solution of equations (1) may be represented in terms of certain harmonic functions for problems involving surfaces of discontinuities or plane cracks. Suppose that the displacements and stresses induced by T are of the symmetric pattern, then

$$\begin{aligned} &u, \sigma_r, \sigma_\theta, \sigma_z; \qquad \text{even in } z\\ &w, \tau_{rz}; \qquad \text{odd in } z \end{aligned} \tag{4}$$

Adopting equations (12) in [3] to the axisymmetric problem under consideration, the displacements u and w can be expressed in terms of two harmonic functions $f(r, z)$ and $\Omega(r, z)$;

$$\begin{aligned} u &= (1-2\nu)\frac{\partial f}{\partial r}+\int_z^\infty \frac{\partial \Omega}{\partial r}\,\mathrm{d}z+z\frac{\partial F}{\partial r},\\ w &= -2(1-\nu)\frac{\partial f}{\partial z}+z\frac{\partial F}{\partial z}, \end{aligned} \tag{5}$$

where

$$F = \Omega+\frac{\partial f}{\partial z},$$

and

$$\nabla^2 f(r, z) = 0, \qquad \nabla^2\Omega(r, z) = 0.$$

The thermoelastic potential $\Omega(r, z)$ can be determined from the temperature field as

$$\frac{\partial\Omega}{\partial z} = \frac{1}{2}\left(\frac{1+\nu}{1-\nu}\right)\alpha T(r, z), \tag{6}$$

and can be associated with the Boussinesq logarithmic potential for a disk whose boundary conforms to that of a crack. At infinity, although the potential $\Omega(r, z)$ is permitted to become unbounded, the regularity condition of the displacement vector requires $\Omega(r, z)$ to have bounded derivatives of all orders with respect to r and z. The limits of integration appearing in the first of equations (5) were chosen to ensure the boundness of u as $z \to \infty$.

* The harmonic-function representation in [3] was developed originally for solving non-axially symmetric problems of plane cracks.

Now, substituting equations (5) into (3) yield the following expressions for the stresses:

$$\frac{\sigma_r}{2\mu} = (1-2\nu)\frac{\partial^2 f}{\partial r^2} - 2\nu\frac{\partial^2 f}{\partial z^2} + \int_z^\infty \frac{\partial^2 \Omega}{\partial r^2}\,dz - 2\frac{\partial \Omega}{\partial z} + z\frac{\partial^2 F}{\partial r^2},$$

$$\frac{\sigma_\theta}{2\mu} = (1-\nu)\frac{1}{r}\frac{\partial f}{\partial r} - 2\nu\frac{\partial^2 f}{\partial z^2} + \frac{1}{r}\int_z^\infty \frac{\partial \Omega}{\partial r}\,dz - 2\frac{\partial \Omega}{\partial z} + z\frac{1}{r}\frac{\partial F}{\partial r},$$

$$\frac{\sigma_z}{2\mu} = -\frac{\partial F}{\partial z} + z\frac{\partial^2 F}{\partial z^2}, \tag{7}$$

$$\frac{\tau_{rz}}{2\mu} = z\frac{\partial^2 F}{\partial r \partial z}, \qquad \tau_{r\theta} = \tau_{\theta z} = 0.$$

Considerations of the evenness and oddness of the displacements and stresses as stated in equations (4) together with the prescribed thermal conditions on the crack surfaces reduce the crack problem to one of an elastic half-space with mixed boundary conditions on the plane $z = 0$. In view of symmetry, the plane $z = 0$ must be free from the shearing stress τ_{rz} and $w(r, 0)$ must vanish inside the circular region $r \leqslant 1$. Without loss in generality, the crack surfaces may be assumed to be free from mechanical loads, i.e. $\sigma_z = 0$ for $r \geqslant 1$ and $z = 0^{\pm}$. The case when the external penny-shaped crack is subjected to surface tractions has already been treated by Lowengrub and Sneddon [7],* and will not be repeated here. Thus, the requisite thermal and elastic boundary conditions on the plane $z = 0$ are taken to be

$$\frac{\partial T}{\partial z} = 0, \qquad 0 \leqslant r < 1,$$

$$T = T(r), \qquad r > 1, \tag{8}$$

and

$$w = 0, \qquad 0 \leqslant r < 1,$$

$$\sigma_z = 0, \qquad r > 1, \tag{9}$$

$$\tau_{rz} = 0, \qquad 0 \leqslant r < \infty.$$

It should be mentioned that the antisymmetric problem in which

$$u, \sigma_r, \sigma_\theta, \sigma_z; \qquad \text{odd in } z$$

$$w, \tau_{rz}; \qquad \text{even in } z \tag{10}$$

may also be formulated by following the procedure of Kassir and Sih [3]. Hence, the two problems, one symmetric and the other antisymmetric, may be superimposed to yield the solution to problems of the infinite solid with any thermal conditions specified on the external penny-shaped crack.

* In what follows, their solution [7] will be added onto that obtained in this paper for computing the size of the plastic zone at the crack boundary.

STEADY-STATE TEMPERATURE DISTRIBUTION

For a semi-infinite solid $z \geqslant 0$ that is free from disturbance at infinity, the appropriate solution of equation (2) is [1]

$$T(r, z) = \int_0^\infty A(s)\,\mathrm{e}^{-sz} J_0(rs)\,\mathrm{d}s, \qquad z \geqslant 0. \tag{11}$$

In equation (11), J_0 is the zero order Bessel function of the first kind and $A(s)$ is a function of the parameter s to be determined from the thermal boundary conditions in equations (8) with $T(r) = T_0 g(r)$, where T_0 is a constant. The function $g(r)$ is to be bounded at infinity and the integral

$$\int_1^\infty g(r)\,\mathrm{d}r,$$

is to be absolutely convergent.

With the help of equation (11), the conditions in equation (8) lead to the dual integral equations

$$\begin{aligned} &\int_0^\infty sA(s)J_0(rs)\,\mathrm{d}s = 0, && 0 \leqslant r < 1 \\ &\int_0^\infty A(s)J_0(rs)\,\mathrm{d}s = T_0 g(r), && r > 1 \end{aligned} \tag{12}$$

which determine the only unknown $A(s)$. The solution of these equations has been given by many previous authors* and can be found in the open literature:

$$A(s) = -T_0\left(\frac{2s}{\pi}\right)^{\frac{1}{2}} \int_1^\infty t^{\frac{1}{2}} J_{\frac{1}{2}}(st)\,\mathrm{d}t\left[\frac{\mathrm{d}}{\mathrm{d}t}\int_t^\infty \frac{rg(r)\,\mathrm{d}r}{\sqrt{(r^2-t^2)}}\right]. \tag{13}$$

Upon defining the function

$$\phi(t) = \frac{2}{\pi}T_0 \int_t^\infty \frac{rg(r)\,\mathrm{d}r}{\sqrt{(r^2-t^2)}}, \tag{14}$$

equation (13) becomes

$$A(s) = -\int_1^\infty \sin(st)\phi'(t)\,\mathrm{d}t, \tag{15}$$

where $\phi'(t) = \mathrm{d}\phi/\mathrm{d}t$. equation (15) may be inserted into equation (11) to give the temperature distribution throughout the solid. However, for the purpose of setting up the mechanical boundary conditions in the subsequent work, it suffices to compute the temperature on the plane $z = 0$. It may be shown that

$$T(r,o) = -\int_{\max(1,r)}^\infty \frac{\phi'(t)\,\mathrm{d}t}{\sqrt{(t^2-r^2)}}, \tag{16}$$

* See for example [8].

in which $\phi'(t)$ can be calculated from equation (14) once $g(r)$ is given. Two examples of interest will be cited.

(1) Consider the problem of heating up the faces of an external circular crack over a ring whose inner and outer radii are unity and a, respectively. In this case, $g(r)$ takes the form

$$g(r) = H(a-r) = \begin{cases} 1, & a > r \\ 0, & a < r \end{cases}, \qquad r > 1 \tag{17}$$

where $H(r)$ represents the Heaviside step function. A straightforward calculation gives

$$\phi(t) = \frac{2}{\pi}T_0\sqrt{(a^2-t^2)}H(a-t), \quad t < a$$

and hence $T(r, 0)$ may be found from equation (16). The result is

$$T(r, o) = \frac{2}{\pi}T_0\int_1^a \frac{t\,\mathrm{d}t}{\sqrt{[(a^2-t^2)(t^2-r^2)]}} = \frac{2}{\pi}T_0\sin^{-1}\left(\frac{a^2-1}{a^2-r^2}\right)^{\frac{1}{2}}, \qquad 0 \leqslant r < 1 \tag{18}$$

and the condition $T(r,0) = T_0$ for $r > 1$ is obviously satisfied.

(2) If the temperature variation on the crack faces is such that

$$g(r) = r^{-n}, \qquad n > 1; \qquad r > 1 \tag{19}$$

then equation (14) yields

$$\phi(t) = \frac{T_0}{\sqrt{\pi}}\frac{\Gamma(n/2-\frac{1}{2})}{\Gamma(n/z)}t^{1-n}, \qquad n > 1$$

where $\Gamma(n)$ is the Gamma function. Putting $\phi(t)$ into equation (16) and carrying out the integration, $T(r,0)$ is obtained:

$$T(r, 0) = \frac{T_0(n-1)\Gamma(n/2-\frac{1}{2})}{\Gamma(n/2)\,2\sqrt{\pi}}r^{-n}B_{r^2}\left(\frac{n}{2}, \frac{1}{2}\right), \qquad 0 \leqslant r < 1; \qquad n > 1. \tag{20}$$

Note that $B_x(m, n)$ is the incomplete Beta function defined by

$$B_x(m, n) = \int_0^x y^{m-1}(1-y)^{n-1}\,\mathrm{d}y, \qquad \mathrm{Re}[m] > 0; \qquad \mathrm{Re}[n] > 0.$$

The complete Beta function $B(m,n)$ may be related to the Gamma functions as

$$B(m, n) = \int_0^1 y^{m-1}(1-y)^{n-1}\,\mathrm{d}y = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

When $n = 2$, 3, etc., the incomplete Beta function in equation (20) reduces to elementary functions:

(a) $n = 2$.

$$T(r, 0) = \frac{T_0}{r^2}[1-(1-r^2)^{\frac{1}{2}}], \qquad 0 \leqslant r < 1$$

(b) $n = 3$.

$$T(r, 0) = \frac{2T_0}{\pi r^3}[\sin^{-1}r - r(1-r^2)^{\frac{1}{2}}], \qquad 0 \leqslant r < 1.$$

Similar expressions of $T(r, 0)$ for other values of n may also be deduced, but they will not be considered here. For $r > 1$, the prescribed temperature distribution of $T(r, 0) = T_0 r^{-n}$ can be easily recovered from equation (16).

Temperature distributions corresponding to other types of thermal boundary conditions are worked out in the Appendix.

THERMAL STRESSES AND DISPLACEMENTS

It is seen from equations (5) and (7) that the evaluation of the displacements and stresses does not warrant an explicit expression of the thermo-elastic potential $\Omega(r, z)$ with respect to r and z.

First of all, equations (6) and (11) may be combined to eliminate $T(r, z)$:

$$\frac{\partial \Omega}{\partial z} = \frac{1}{2}\left(\frac{1+\nu}{1+\nu}\right)\alpha \int_0^\infty A(s)\, \mathrm{e}^{-sz} J_0(rs)\, \mathrm{d}a, \qquad z \geqslant 0, \tag{21}$$

from which

$$\frac{\partial \Omega}{\partial r} = \frac{1}{2}\left(\frac{1+\nu}{1-\nu}\right)\alpha \int_0^\infty A(s)\, \mathrm{e}^{-sz} J_1(rs)\, \mathrm{d}s, \qquad z \geqslant 0, \tag{22}$$

is obtained. The arbitrary function of integration may be set to zero, since $\partial\Omega/\partial r$ must vanish in the limit as $z \to \infty$. Equation (22) may be integrated with respect to z giving

$$\int_z^\infty \frac{\partial \Omega}{\partial r}\, \mathrm{d}z = \frac{1}{2}\left(\frac{1+\nu}{1-\nu}\right)\alpha \int_0^\infty \frac{1}{s} A(s) \mathrm{e}^{-sz} J_1(rs)\, \mathrm{d}s, \qquad z \geqslant 0. \tag{23}$$

Having determined the temperature field $T(r, z)$ or $A(s)$ for various prescribed thermal conditions, it is clear that the quantities

$$\frac{\partial \Omega}{\partial z}, \quad \frac{\partial \Omega}{\partial r}, \quad \int_z^\infty \frac{\partial \Omega}{\partial r}\, \mathrm{d}z, \text{ etc.}$$

appearing in equations (5) and (7) can be calculated in a straightforward manner.

It is now more pertinent to find the unknown harmonic function $\partial f/\partial z$ from the remaining mechanical boundary conditions in equations (9). A quick glance at equations (7) reveals that on the plane $z = 0$, τ_{rz} vanishes automatically and the remaining two conditions in equations (9) require that

$$\begin{aligned} \frac{\partial f}{\partial z} &= 0, \qquad 0 \leqslant r < 1 \\ \frac{\partial^2 f}{\partial z^2} &= -\frac{1}{2}\left(\frac{1+\nu}{1-\nu}\right)\alpha T(r, 0), \qquad r > 1 \end{aligned} \tag{24}$$

where

$$\frac{\partial f}{\partial z}, \quad \frac{\partial^2 f}{\partial z^2} \to 0, \qquad \text{as} \qquad z \to \infty.$$

Taking into account the axisymmetric nature of the thermal loading, $\partial f/\partial z$ may be represented by the Hankel integral

$$\frac{\partial f}{\partial z} = \int_0^\infty \frac{1}{s} B(s)\, e^{-sz} J_0(rs)\, ds, \qquad z \geqslant 0. \tag{25}$$

By virtue of equations (24), the function $B(s)$ has to be found from the pair of simultaneous equations

$$\begin{aligned} &\int_0^\infty \frac{1}{s} B(s) J_0(rs)\, ds = 0, \qquad 0 \leqslant r < 1 \\ &\int_0^\infty B(s) J_0(rs)\, ds = \frac{1}{2}\left(\frac{1+\nu}{1-\nu}\right)\alpha T(r), \qquad r < 1 \end{aligned} \tag{26}$$

in which $T(r)$ represents the axisymmetric temperature variation prescribed on the plane $z = 0$. Lowengrub and Sneddon [8] and others have shown that the satisfaction of equations (26) can be achieved by expressing $B(s)$ in terms of the function

$$\psi(t) = \left(\frac{1+\nu}{1-\nu}\right)\frac{\alpha}{\pi}\int_t^\infty \frac{\eta T(\eta)\, d\eta}{\sqrt{(\eta^2 - t^2)}}, \tag{27}$$

through an integral of the form

$$B(s) = s\int_1^\infty \psi(t)\cos(st)\, dt. \tag{28}$$

With a knowledge of $B(s)$, the problem of determining the displacements and stresses in the elastic solid is reduced to quadrature.

For the purpose of finding the displacements on the crack surfaces, it may be shown that for $z = 0$

$$\frac{\partial f}{\partial z} = \begin{cases} \displaystyle\int_1^r \frac{\psi(t)\, dt}{\sqrt{(r^2 - t^2)}}, & r > 1 \\ 0, & 0 \leqslant r < 1 \end{cases} \tag{29}$$

and

$$\frac{\partial f}{dr} = \frac{1}{r}\begin{cases} \displaystyle\int_1^\infty \psi(t)\, dt - \int_r^\infty \frac{t\psi(t)\, dt}{\sqrt{(t^2 - r^2)}}, & r > 1 \\ \displaystyle\int_1^\infty [1 - t(t^2 - r^2)^{-\frac{1}{2}}]\psi(t)\, dt, & 0 \leqslant r < 1. \end{cases} \tag{30}$$

Hence, equations (23), (29) and (30) may be substituted into equation (5) and the resulting expression for the displacement on the crack plane are

$$\begin{aligned} u(r,0) &= \frac{1}{2}\left(\frac{1+\nu}{1-\nu}\right)\alpha\int_0^\infty \frac{1}{s} A(s) J_1(rs)\, ds + (1-2\nu)\frac{1}{r}\left[\int_1^\infty \psi(t)\, dt - \int_r^\infty \frac{t\psi(t)\, dt}{\sqrt{(t^2 - r^2)}}\right], \qquad r > 1 \\ w(r,0) &= -2(1-\nu)\int_1^r \frac{\psi(t)\, dt}{\sqrt{(r^2 - t^2)}}, \qquad r > 1. \end{aligned} \tag{31}$$

Similarly, the displacements u and w for points inside the circular region of unit radius can be found:

$$u(r,0) = \frac{1}{2}\left(\frac{1+v}{1-v}\right)\alpha\int_0^\infty \frac{1}{s}A(s)J_1(rs)\,\mathrm{d}s + (1-2v)\frac{1}{r}\int_0^\infty [1-t(t^2-r^2)^{-\frac{1}{2}}]\psi(t)\,\mathrm{d}t, \qquad 0 \leqslant r < 1$$

$$w(r,0) = 0, \qquad 0 \leqslant r < 1 \tag{32}$$

The functions $A(s)$ and $\psi(t)$ in the above expressions are defined by equations (15) and (27), respectively.

Of particular interest is the stress component σ_z from which the crack-border stress–intensity factor formula may be extracted. This factor has been known to control the instability behavior of cracks in the theory of brittle fracture [5]. To this end, equations (6), (25) and (28) are substituted into the third of equations (7) and hence for $z = 0$ σ_z becomes

$$\sigma_z(r,0) = 2\mu\left\{\int_0^\infty \left[s\int_1^\infty \psi(t)\cos(st)\,\mathrm{d}t\right]J_0(rs)\,\mathrm{d}s - \frac{1}{2}\left(\frac{1+v}{1-v}\right)\alpha T(r)\right\}.$$

Therefore, it is not difficult to show that

$$\sigma_z(r,0) = -2\mu\left[\frac{\psi(r)}{\sqrt{(1-r^2)}} + \int_1^\infty \frac{\psi'(t)\,\mathrm{d}t}{\sqrt{(t^2-r^2)}} + \frac{1}{2}\left(\frac{1+v}{1-v}\right)\alpha T(r)\right], \qquad 0 \leqslant r < 1 \tag{33}$$

and $\sigma_z(r, 0) = 0$ for $r > 1$. Notice that only the leading term in equation (33) contributes to the singular behavior of σ_z, while the other two expressions are bounded as $r \to 1$. Thus, by letting $\varepsilon = 1-r$ and $\varepsilon \to 0$, σ_z becomes

$$\sigma_z(r,0) = -\frac{2\mu\psi(1)}{\sqrt{(2\varepsilon)}} + \ldots$$

where terms of order higher than $\varepsilon^{-\frac{1}{2}}$ have been dropped. The coefficient of $1/\sqrt{(2\varepsilon)}$, say k_1. is the crack–border stress–intensity factor for the opening mode of crack extension, i.e.

$$k_1 = -2\mu\psi(1) = -\left(\frac{1+v}{1-v}\right)\frac{2\mu\alpha}{\pi}\int_1^\infty \frac{\eta T(\eta)\,\mathrm{d}\eta}{\sqrt{(\eta^2-1)}}. \tag{34}$$

By the same procedure, the other stress components may also be expressed in terms of $T(r)$, the prescribed temperature distribution on the crack.

EXTERNAL CRACK AROUND INSULATED CIRCULAR REGION

To fix ideas, the displacements and stresses on the plane $z = 0$ corresponding to the temperature distribution discussed earlier may be expressed explicitly in terms of

$T(r) = T_0 g(r)$. Appropriate elimination of $A(s)$ and $\psi(t)$ in equations (31) and (32) gives the displacement field.

$$u(r,0) = \left(\frac{1+\nu}{1-\nu}\right)\frac{\alpha T_0}{\pi r}\begin{cases} (1-\sqrt{(1-r^2)})\displaystyle\int_1^\infty \frac{\eta g(\eta)\,d\eta}{\sqrt{(\eta^2-1)}} + 2(1-\nu)\int_1^\infty \left[1-\frac{t}{\sqrt{(t^2-r^2)}}\right] \\ \qquad \left[\displaystyle\int_t^\infty \frac{\eta g(\eta)\,d\eta}{\sqrt{(\eta^2-t^2)}}\right] dt, \quad 0 \leqslant r < 1 \\ \displaystyle\int_1^\infty \frac{\eta g(\eta)\,d\eta}{\sqrt{(\eta^2-1)}} + (1-\nu)\left[2\int_1^\infty dt \int_t^\infty \frac{\eta g(\eta)\,d\eta}{\sqrt{(\eta^2-t^2)}} \right. \\ \qquad \left. -\pi \displaystyle\int_r^\infty \eta g(\eta)\,d\eta\right], \quad r > 1 \end{cases} \tag{35}$$

and

$$w(r,0) = -\frac{2(1+\nu)\alpha T_0}{\pi}\begin{cases} 0, & 0 \leqslant r < 1 \\ \displaystyle\int_1^r \left[\int_1^\infty \frac{\eta g(\eta)\,d\eta}{\sqrt{(\eta^2-t^2)}}\right]\frac{dt}{\sqrt{(r^2-t^2)}}, & r > 1 \end{cases} \tag{36}$$

Equation (27) may be combined with equation (33) to put the normal stress component in the form

$$\sigma_z(r,0) = -\frac{E\alpha T_0}{(1-\nu)\pi}(1-r^2)^{-\frac{1}{2}}\begin{cases} \displaystyle\int_1^\infty \frac{\eta g(\eta)\,d\eta}{\sqrt{(\eta^2-1)}}, & 0 \leqslant r < 1 \\ 0, & r > 1 \end{cases} \tag{37}$$

where E is Young's modulus of elasticity. In deriving equation (37), it is interesting to note that the two non-singular terms in equation (33) cancelled each other.

Let $g(r)$ in equations (35) to (37) be given by equations (17) and (19). The calculation of u, w and σ_z involves a considerable amount of detailed work which will be omitted. The final results are:

(1) *Step function*

In this case, the displacement component in the radial direction is

$$u(r,0) = \left(\frac{1+\nu}{1-\nu}\right)\frac{\alpha T_0}{\pi r}\begin{cases} \nu\sqrt{(a^2-1)}(1-\sqrt{(1-r^2)}) + (1-\nu)a^2\left[\sin^{-1}\left(\dfrac{a^2-1}{a^2}\right)^{\frac{1}{2}} \right. \\ \qquad \left. -\left(1-\dfrac{r^2}{a^2}\right)\sin^{-1}\left(\dfrac{a^2-1}{a^2-r^2}\right)^{\frac{1}{2}}\right], \quad 0 \leqslant r < 1 \\ \nu\sqrt{(a^2-1)} + (1-\nu)a^2\left[\dfrac{\pi}{2}\left(\dfrac{r}{a}\right)^2 - \sin^{-1}\left(\dfrac{1}{a}\right)\right], \quad 1 < r < a \\ \nu\sqrt{(a^2-1)} + (1-\nu)a^2\sin^{-1}\left(\dfrac{a^2-1}{a^2}\right)^{\frac{1}{2}}, \quad r > a \end{cases} \tag{38}$$

and the normal displacement component is given by

$$w(r,0) = -\frac{2(1+v)\alpha r T_0}{\pi}\begin{cases} 0, & 0 \leqslant r < 1 \\ \dfrac{a}{r}\left[E\left(\dfrac{r}{a},\dfrac{\pi}{2}\right)-E\left(\dfrac{r}{a},\alpha_1\right)\right], & 1 < r < a \\ E\left(\dfrac{a}{r},\dfrac{\pi}{2}\right)-E\left(\dfrac{a}{r},\alpha_2\right)-\left(1-\dfrac{a^2}{r^2}\right)\left[K\left(\dfrac{a}{r},\dfrac{\pi}{2}\right)-K\left(\dfrac{a}{r},\alpha_2\right)\right], & r > a \end{cases} \tag{39}$$

in which $E(r/a, \alpha_1)$ and $K(a/r, \alpha_2)$ are the incomplete elliptic integrals of the second and first kind, respectively, where

$$\alpha_1 = \sin^{-1}\left(\frac{1}{r}\right), \quad 0 < \alpha_1 < \frac{\pi}{2} \quad \text{and} \quad \alpha_2 = \sin^{-1}\left(\frac{1}{a}\right), \quad 0 < \alpha_2 < \frac{\pi}{2}$$

When α_1 or $\alpha_2 \to \pi/2$, E and K become the complete elliptic integrals. The normal stress component is

$$\sigma_z(r,0) = -\frac{E\alpha T_0}{(1-v)\pi}(1-r^2)^{-\frac{1}{2}}\begin{cases} \sqrt{(a^2-1)}, & 0 \leqslant r < 1 \\ 0, & r > 1 \end{cases} \tag{40}$$

and it follows that

$$k_1 = -\frac{E\alpha T_0}{(1-v)\pi}\sqrt{(a^2-1)}. \tag{41}$$

(2) *Radial decay*

If the temperature on the crack varies in accordance with equation (19), then for $n > 2$

$$u(r,0) = \frac{1}{2}\left(\frac{1+v}{1-v}\right)\frac{\alpha T_0 \Gamma[(n/2)-\frac{1}{2}]}{\sqrt{(\pi)} r \Gamma(n/2)}$$

$$\begin{cases} 1-\sqrt{(1-r^2)}+(1+v)\left[\dfrac{2}{n-2}-r^{2-n}B_{r^2}\left(\dfrac{n}{2}-1,\dfrac{1}{2}\right)\right], & 0 \leqslant r < 1 \\ 1+\dfrac{2(1-v)}{n-2}\left[1-\dfrac{\sqrt{(\pi)}\Gamma(n/2)}{\Gamma[(n-1)/2]}r^{2-n}\right], & r > 1 \end{cases} \tag{42}$$

and for $n > 1$

$$w(r,0) = -\frac{(1+v)\alpha T_0 \Gamma[(n/2)-\frac{1}{2}]}{2\sqrt{(\pi)}\Gamma(n/2)}\begin{cases} 0, & 0 \leqslant r < 1 \\ r^{1-n}B_{1-r^{-2}}[\frac{1}{2}, 1-(n/2)], & r > 1 \end{cases} \tag{43}$$

For $g(r) = r^{-n}$ and $n > 1$, equation (37) reduces to

$$\sigma_z(r,0) = -\frac{E\alpha T_0}{(1-v)\pi}(1-r^2)^{-\frac{1}{2}}\begin{cases} \dfrac{\sqrt{(\pi)}}{2}\dfrac{\Gamma[(n/2)-\frac{1}{2}]}{\Gamma(n/2)}, & 0 \leqslant r < 1 \\ 0, & r > 1 \end{cases} \tag{44}$$

Therefore, the k_1-factor is obtained:

$$k_1 = -\frac{E\alpha T_0 \Gamma[(n/2)-\frac{1}{2}]}{2(1-v)\sqrt{(\pi)}\Gamma(n/2)}, \quad n > 1 \tag{45}$$

To recapitulate, the stress-intensity factors given in equations (41) and (45) can be associated with the forces which motivate and produce crack extension owing to thermal disturbances. The critical values of k_1 for a particular material can usually be measured experimentally. Moreover, if the material undergoes plastic yielding at the crack border, where the thermal stresses are exceedingly high, there will be a localized zone of plasticity surrounding the periphery of the crack. The size of this plastic zone for an external penny-shaped crack will be estimated in the next section.

THERMAL PLASTIC ZONE SIZE

An ideal elastic–plastic model for the plane extension problem of a straight crack in a thin sheet has been proposed by Dugdale [6]. This model will be adopted to estimate the extent of plastic yielding at the edge of an external circular crack. The material near the crack is assumed to flow after yielding at a constant tensile stress q_0 and the plastic zone is confined to a thin layer of width ω around the uncracked portion of the plane $z = 0$. The parameter ω will be determined from the finiteness condition of σ_z at the leading edge of the plastic zone.

Mathematically, the solid may be assumed to deform elastically under the action of thermal loading together with a mechanical compressive stress, $-q_0$, distributed over the surface of a ring of inner radius $r = 1$ and outer radius $r = 1+\omega$. For this problem, σ_z can be obtained by superimposing the solution of Lowengrub and Sneddon [7] onto that of equation (37). The normal stress component for the combined thermal and mechanical problem is

$$\sigma_z(r,0) = \begin{cases} \dfrac{E}{\sqrt{(1-r^2)}}\left[-\dfrac{\alpha T_0}{(1-\nu)\pi}\displaystyle\int_1^\infty \dfrac{\eta g(\eta)\,\mathrm{d}\eta}{\sqrt{(\eta^2-t^2)}}+\dfrac{\phi_1(1)}{1+\nu}\right]+\dfrac{E}{1+\nu}\displaystyle\int_1^\infty \dfrac{\phi_1'(t)\,\mathrm{d}t}{\sqrt{(t^2-r^2)}}, & 0 \leqslant r < 1 \\ -p(r), \quad r > 1 & \end{cases} \tag{46}$$

where

$$\phi_1(t) = \frac{1}{\pi\mu}\int_t^\infty \frac{\xi p(\xi)\,\mathrm{d}\xi}{\sqrt{(\xi^2-t^2)}},$$

and

$$p(r) = +g_0 H(1+\omega-r).$$

Since σ_z is to be bounded at $r = 1$, the singular terms in equation (46) must be removed by taking

$$\alpha\mu T_0 \int_1^\infty \frac{\eta g(\eta)\,\mathrm{d}\eta}{\sqrt{(\eta^2-t^2)}} = \left(\frac{1-\nu}{1+\nu}\right)\int_1^\infty \frac{\xi p(\xi)\,\mathrm{d}\xi}{\sqrt{(\xi^2-t^2)}}.$$

Setting $2\mu(1+\nu) = E$ and performing the integration with respect to ξ lead to the equation

$$E\alpha T_0 \int_1^\infty \frac{\eta g(\eta)\,\mathrm{d}\eta}{\sqrt{(\eta^2-t^2)}} = +2(1-\nu)q_0\sqrt{[\omega(\omega+2)]}, \tag{47}$$

for evaluating the plastic zone size ω. For illustration, formulas for ω are worked out for the two previously mentioned examples.

(1) If $g(r) = H(1+\beta-r)$, then equation (47) may be integrated and solved for ω:

$$\omega = -1+[1+\beta(\beta+2)\gamma^2]^{\frac{1}{2}}. \tag{48}$$

The quantity

$$\gamma = +\frac{E\alpha T_0}{2(1-\nu)q_0},$$

may be interpreted as the ratio of the applied thermal stress to the yield stress of the material q_0, and β in equation (48) is the width of the region heated by the constant temperature T_0. A plot of ω vs. γ for various values of β are shown in Fig. 1. The curves are similar in trend to that found by Dugdale [6] for the two-dimensional problem of an isothermal crack.

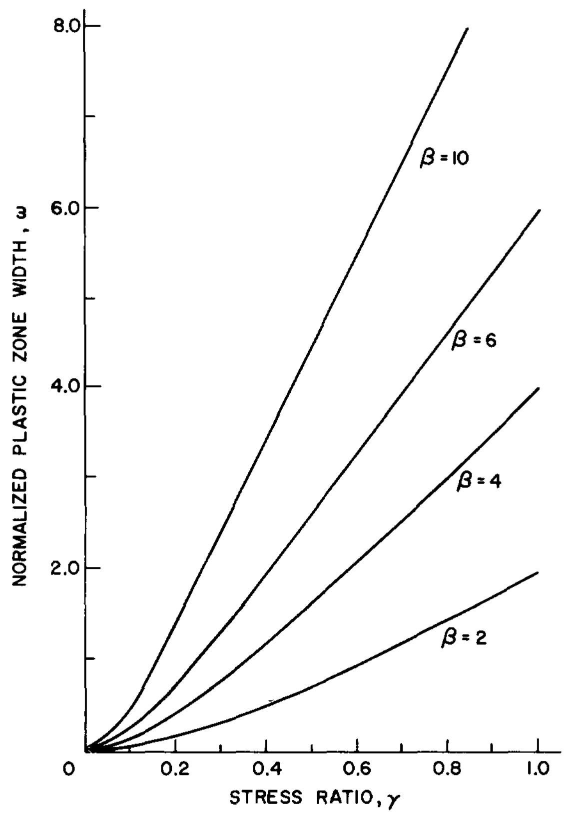

FIG. 1. Widths of plastic zone for constant heating.

(2) For $g(r) = r^{-n}$ with $n > 1$, the plastic zone size is found to be

$$\omega = -1+\left\{1+\frac{\pi}{4}\left[\gamma\frac{\Gamma[(n/2)-\frac{1}{2}]}{\Gamma(n/2)}\right]^2\right\}^{\frac{1}{2}}, \tag{49}$$

whose variations with γ for different values of n are plotted in Fig. 2. As to be expected, the size of the plastic zone increases as the temperature T_0 is raised.

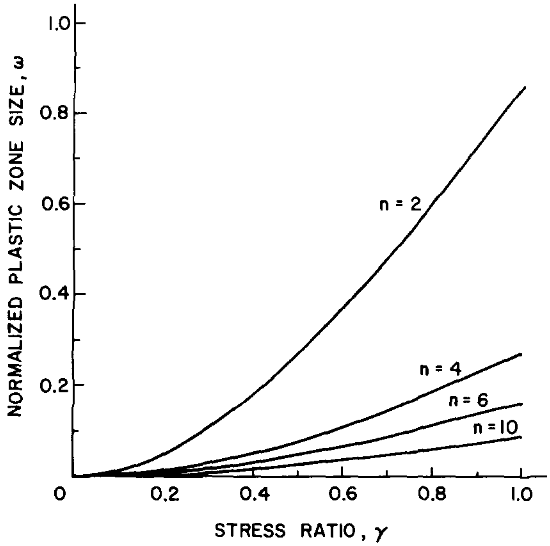

FIG. 2. Widths of plastic zone for temperature decaying radially.

DISPLACEMENTS NORMAL TO CRACK SURFACES

At the leading edge of the external circular crack, the tangent to the normal displacement $w(r, 0)$ coincides with the uncracked portion of the plane $z = 0$. In other words, the surfaces of the crack close smoothly as predicted in the Barenblatt model [10]. This may be verified for the two examples discussed earlier.

(1) *Step function*

From equation (39) and equations (4.7) and (5.4) in [7], it is found that

$$w(r,0) = \frac{2}{\pi}(1+\nu)\left\{-\alpha(1+\beta)T_0\left[E\left(\frac{r}{1+\beta},\frac{\pi}{2}\right)-E\left(\frac{r}{1+\beta},\alpha_1\right)\right]\right.$$
$$\left.+2(1-\nu)(1+\omega)\frac{q_0}{E}\left[E\left(\frac{r}{1+\omega},\frac{\pi}{2}\right)-E\left(\frac{r}{1+\omega},\alpha_1\right)\right]\right\},\qquad 1 < r < \alpha$$

Hence, as $r \to 1$ the result

$$\left(\frac{\partial w}{\partial r}\right)_{z=0} = \frac{2(1+\nu)}{\pi E r\sqrt{(r^2-1)}}[2(1-\nu)q_0\sqrt{\{\omega(\omega+2)\}}-E\alpha T_0\sqrt{\{\beta(\beta+2)\}}]+\ldots$$

is obtained. The quantity between the brackets vanish by virtue of equation (48) which determines the size of the plastic zone. This implies that the crack surfaces close smoothly.

(2) *Radial decay*

Similarly, using equation (43) in conjunction with equations (4.7) and (5.4) in [7] lead to

$$\left(\frac{\partial w}{\partial r}\right)_{z=0} = \frac{1+\nu}{r\sqrt{(r^2-1)}}\left[\frac{4(1-\nu)}{\pi E}\sqrt{\{\omega(\omega+2)\}}-\frac{\alpha T_0\Gamma[(n/2)-\frac{1}{2}]}{\sqrt{\pi}\Gamma(n/2)}\right]+\ldots$$

Because of equation (49), the above expression vanishes in the limit as $r \to 1$.

PLASTIC ENERGY DISSIPATION

The plastic energy dissipation around the leading edge of the external crack may be calculated by assuming the radius of the uncracked portion of the solid ($z = 0, 0 \leqslant r < 1$) decrease slowly under a constant value of γ. This decrease in the radius is accompanied by a small change $\Delta\omega$ in the width of plastic zone ω such that

$$\frac{\Delta\omega}{\Delta R} = \frac{\omega}{R}$$

The quantity R is the radius of the uncracked portion of the solid which has been taken to be unity in the present analysis.

By neglecting terms of order higher than the first in dR the plastic dissipation is given by

$$\tfrac{1}{2}\,\mathrm{d}W_p = q_0 \int_{r=R}^{r=R+\omega} \left(\frac{\partial W}{\partial R}\right)_{z=0} 2\pi r\,\mathrm{d}r$$

Now, the displacement $w(r, 0)$ can be expressed in terms of R and the above integral may be calculated in a straightforward manner to yield the rate of energy dissipation per unit area of the new surface of the external crack.

CONCLUDING REMARKS

The linear thermoelastic problem of an elastic solid containing an external penny-shaped crack has been formulated and solved. The temperature and/or heat flux can be applied either symmetrically or antisymmetrically with respect to the plane in which the crack occupies. The solution offers the possibility of a theory of brittle fracture for crack propagation caused by heating. This can be verified by experimentally measuring the critical values of the stress–intensity factors as proposed earlier.

The obtained displacement field also permits an evaluation of the plastic energy dissipation for cracking induced by thermal stresses.

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APPENDIX

Temperature fields pertaining to thermal boundary conditions not covered in the text will be presented below.

Case A. Instead of applying temperature to the crack, heat flux may be specified on the flat surfaces $r > 1$ and $z = 0^{\pm}$. The distribution of temperature that satisfies the set of conditions

$$\frac{\partial T}{\partial z} = \begin{cases} 0, & 0 \leqslant r < 1 \\ Q(r), & r > 1 \end{cases} \tag{50}$$

is given by

$$T(r, z) = -\int_0^{\infty} \left[\int_1^{\infty} \eta Q(\eta) J_0(s\eta)\, \mathrm{d}\eta\right] \mathrm{e}^{-sz} J_0(rs)\, \mathrm{d}s, \qquad z \geqslant 0 \tag{51}$$

Consider two special cases of $Q(r)$:

(1) Suppose that $Q(r) = Q_0 H(a-r)$, where Q_0 is a constant. Then equation (51) may be simplified to integrals of the Lipschitz–Hankel type

$$T(r, z) = Q_0 \int_0^{\infty} \frac{1}{s}[J_1(s) - aJ_1(as)]\, \mathrm{e}^{-sz} J_0(rs)\, \mathrm{d}s, \qquad a > 1 \tag{52}$$

Such integrals have been evaluated and tabulated numerically in [9].

(2) In the case, when $Q(r) = Q_0 r^{-n}$ with $n > 1$, the temperature field is

$$T(r, z) = -Q_0 \int_0^{\infty} I_0(n, s)\, \mathrm{e}^{-sz} J_0(rs)\, \mathrm{d}s \tag{53}$$

where

$$I_0(n, s) = \int_1^{\infty} \xi^{1-n} J_0(s\xi)\, \mathrm{d}\xi, \qquad n > 1$$

For $z = 0$, it may be shown that

$$T(r, 0) = Q_0 \begin{cases} \dfrac{1}{1-n}\, {}_3F_2(\tfrac{1}{2}, \tfrac{1}{2}, n/2 - \tfrac{1}{2}; 1, 1; r^2), & 0 < r < 1; \quad n > 1 \\ \dfrac{1}{(2-n)r}[{}_3F_2(\tfrac{1}{2}, \tfrac{1}{2}, 1 - n/2; 1, 1; 1/r^2) - r^{2-n}\, {}_3F_2(\tfrac{1}{2}, \tfrac{1}{2}, 1 - n/2; 1, 2 - n/2; 1)], & \\ \qquad r > 1; \quad 1 < n < 2 & \end{cases} \tag{54}$$

where ${}_mF_n(a, b; c; x)$ is the generalized hypergeometric function. The second of equations (54) is valid only for values of n between 1 and 2 and hence it is somewhat limited in application. For other values of n, equation (53) may be used.

Case B. Another possible case is when the uncracked region $0 \leqslant r \leqslant 1$ and $z = 0$ is permitted to conduct heat such that

$$T = \begin{cases} 0, & 0 \leqslant r < 1 \\ T_0 h(r), & r > 1 \end{cases} \tag{55}$$

and thus

$$T(r,z) = T_0 \int_0^\infty \left[\int_1^\infty \eta h(\eta) J_0(s\eta)\, d\eta\right] s\, e^{-sz} J_0(rs)\, ds, \qquad z \geqslant 0 \tag{56}$$

(1) For $h(r) = H(a-r)$, equation (56) is expressible in terms of the Lipschitz–Hankel type of integrals

$$T(r,z) = T_0\left[a\int_0^\infty e^{-sz} J_1(as) J_0(rs)\, ds - \int_0^\infty e^{-sz} J_1(s) J_0(rs)\, ds\right], \qquad z \geqslant 0 \tag{57}$$

which are evaluated in [9].

(2) If $h(r) = r^{-n}$ and $n > 1$, the temperature field becomes

$$T(r,z) = T_0 \int_0^\infty [nI_1(m,s) - J_1(s)]\, e^{-sz} J_0(rs)\, ds, \qquad z \geqslant 0 \tag{58}$$

where

$$I_1(m,s) = \int_1^\infty \xi^{-m} J_1(s\xi)\, d\xi.$$

(*Received* 1 *April* 1968; *revised* 1 *August* 1968)

Абстракт—Решаются задачи линейной термоупругости для полей термических напряжений и перемещений в упругом бесконечном теле, ослабленным плоскостью разрыва или трещиной, занимающей пространство вне круглой области. Поверхности трещины нагреты некоторой удерживающейся температурой или некоторым заданным потоком тепла, которого величина распределения уменьшается в бесконечности. Обращается специальное внимание на случай, когда круглая область, окружена внешней трещиной, изолированна от потока тепла. Решение задачи термических напряжений можно связать с задачей, которая принимает соответствующие силы сцепления на поверхностях трещины и таким образом дающей элементы для расширения гипозы Дюгдапа на термически напряженные тела, содержающие трещины. Более детально, результаты этого анализа дают возможность оценить пределы пластической эоны и диссипации пластической энергии, для внешних, круглых трещин. Заметка зтого рода позваляет понятв механику момента разрушения в пластических материалах.